Abstract

This is a set of notes for lectures which I am giving at TASI2011, provided for the benefit of the students there. The notes provide an introduction to next-to-leading order calculations in QCD and develop the spinor calculus necessary for the computation of both tree-level and one-loop amplitudes (using analytic unitarity techniques). A large part of these notes is based on lectures previously delivered by Keith Ellis in Würzburg and Calcutta. If the notes are prepared for wider dissemination then the incomplete referencing to the literature will be rectified.

Lecture 2
Spinor calculations at tree level

In this section we demonstrate the use of spinor identities in simplifying amplitude calculations. For reference, expressions for Feynman diagram propagators and vertices in QCD are shown in Figure 1.1 and the simplest rules for the electroweak theory (i.e. the rules for triple and quartic boson couplings are omitted) are shown in Figure 1.2.

1.1 $c \bar{s} \rightarrow W^+ \rightarrow \nu e^+$

Consider the process,
\[ c \bar{s} \rightarrow W^+ \rightarrow \nu e^+ \]  
which is represented by the single Feynman diagram shown in Figure 1.3. Arrows represent the fermion flow and all momenta are outgoing. The amplitude is given by,
\[ M = \left( -\frac{ig_W}{\sqrt{2}} \right)^2 \frac{(-i)}{P_w(s_{ew})} \langle \nu | \gamma^\mu | e \rangle \langle e | \gamma^\nu | \bar{\nu} \rangle \langle \bar{\nu} | \gamma^\rho \gamma^\lambda | c \rangle \equiv \frac{ig_W^2}{P_W(s_{ew})} \langle \nu s | c e \rangle, \]

where \( P_X(p) = p^2 - m_X^2 + im_X \Gamma_X \) and that is the answer. We have assumed that all fermions, including the charmed quark, are massless. The result is particularly simple in this case because of course the $W$ coupling limits us to a single allowed helicity combination.

Compare the calculation performed using the traditional method with traces. The matrix element is given by,
\[ M \propto \bar{u}(\nu \gamma^\alpha \gamma_L u(e)) \bar{u}(s) \gamma_\alpha \gamma_L u(c) \]  
\[ \left| M \right|^2 = \text{Tr} \left\{ \gamma^\alpha \gamma_L \{ \gamma^\beta \gamma_L \gamma^\gamma \gamma_L \} \right\} \left( \gamma_\alpha \gamma_L \{ \gamma^\beta \gamma_L \gamma^\gamma \gamma_L \} \right) \]
\[ = 4 \left\{ \nu^\alpha e^\beta + \nu^\beta e^\alpha - g^{\alpha \beta} e \cdot s + i \epsilon^{\alpha \beta \gamma \delta} \nu_\gamma \nu_\delta \right\} \]
\[ \times \left\{ s_\alpha c_\beta + s_\beta c_\alpha - g_{\alpha \beta} c : s + i \epsilon_{\alpha \beta \gamma \sigma} s^\gamma c^\sigma \right\} \]
\[ = 4 e \cdot c s : \nu \]  

where we have used,
\[ \epsilon^{\alpha \beta \gamma \delta} \epsilon_{\alpha \beta \gamma \delta} = -2 \left[ g_\rho g^\delta - g_\delta g^\rho \right] \]

Using the spinor method the $\gamma$-matrix algebra simply disappears.

1.2 Top production and decay

We now turn to a more complicated case, that of top production and decay from a $q \bar{q}$ initial state. The single diagram for this process is shown in Figure 1.4. The expression for the amplitude, with a left-handed initial quark line, is:
\[ M = \frac{i g_w^4 g^2}{4 D} (t^A)_{\nu \gamma \mu} \frac{1}{D} (t^A)_{\nu \gamma \mu} \langle \nu | \gamma^\mu | e \rangle \langle e | \gamma^\nu | \bar{\nu} \rangle \langle 2 | \gamma^\alpha | 3 \rangle \]
\[ \times \bar{u}(b) \gamma_R \gamma_\mu (p_1 + m_t) \gamma_\alpha (-p_4 + m_t) \gamma_\nu \gamma_L v(\bar{b}) \]  

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Figure 1.1: Feynman rules for QCD.

\[ \delta^{AB} \left[ -g^{\alpha\beta} + (1 - \lambda) \frac{P^\alpha P^\beta}{p^2 + i\epsilon} \right] \frac{i}{p^2 + i\epsilon} \]

\[ \delta^{Aa} \frac{i}{(p^2 + i\epsilon)} \]

\[ \delta^{ab} \frac{i}{(p' - m + i\epsilon)_{ji}} \]

\[ -g f^{ABC} [(p - q)^\gamma g^{\alpha\beta} + (q - r)^\alpha g^{\beta\gamma} + (r - p)^\beta g^{\gamma\alpha}] \]

(all momenta incoming)

\[ -ig^2 f^{XAC} f^{XBD} \left[ g^{\alpha\beta} g^{\gamma\delta} - g^{\alpha\delta} g^{\beta\gamma} \right] \]

\[ -ig^2 f^{XAD} f^{XBC} \left[ g^{\alpha\beta} g^{\gamma\delta} - g^{\alpha\gamma} g^{\beta\delta} \right] \]

\[ -ig^2 f^{XAB} f^{XCD} \left[ g^{\alpha\gamma} g^{\beta\delta} - g^{\alpha\delta} g^{\beta\gamma} \right] \]

\[ g f^{ABC} q^a \]

\[ -ig (t^A)_{cb} (\gamma^a)_{ji} \]
Figure 1.2: The simplest Feynman rules for the electroweak theory (rules for the triple and quartic boson couplings are not shown) in the unitary gauge.
Figure 1.3: Feynman diagram for the process $c\bar{s} \rightarrow W^+ \rightarrow \nu e^+$ or, equivalently, $ce \rightarrow \nu s$.

Figure 1.4: Feynman diagram for the process $q\bar{q} \rightarrow t(\rightarrow \nu e^+ b)\bar{t}(\rightarrow e^-\bar{\nu}\bar{b})$.

where the denominator factor $D$ is given by,

$$D = P_W(\nu + \bar{e}) P_W(\bar{\nu} + e) P_t(p_1) P_t(p_4) s_{23}. \quad (1.7)$$

We can factor out the colour matrices $(t^A)_{ij}$ (normalized such that $\text{Tr}(\tau^A\tau^B) = \delta^{AB}$), overall coupling constants and denominator,

$$\mathcal{M} = g_4^4 g_s^2 \frac{1}{D} (t^A)_{i_2 i_3} (t^A)_{i_1 i_4} \mathcal{M}', \quad (1.8)$$

and use the Fierz identity twice to find,

$$\mathcal{M}' = \frac{1}{2} \langle 2|\gamma^\alpha|3 \rangle \bar{u}(b)|\nu\rangle \langle\bar{e}|(p_1 + m_t)\gamma_\alpha(-p_4 + m_t)|e\rangle \langle\bar{\nu}|v(b). \quad (1.9)$$

Note that, at this point, the $b$-quarks have not been treated as massless – if they were then, for instance, we could make the replacement $\langle\bar{\nu}^+|v(b) \rightarrow [\bar{\nu} b]$. If we further create massless vectors $\hat{p}_1, \hat{p}_4$ out of $p_1$ and $p_4$ by subtracting pieces proportional to the vectors $\bar{e}$ and $e$ respectively,

$$p_1 = \hat{p}_1 + \left(\frac{m_1^2}{2\hat{p}_1 \cdot \bar{e}}\right) \bar{e} \quad (1.10)$$

$$p_4 = \hat{p}_4 + \left(\frac{m_2^2}{2\hat{p}_4 \cdot e}\right) e \quad (1.11)$$
then we can further simplify this to read,
\[ M' = \sqrt{2b \cdot \nu} \sqrt{2\bar{b} \cdot \bar{\nu}} \left\{ -[\bar{e} \hat{1}][\bar{1} 2][\bar{3} \hat{4}][\hat{4} e] + m_1^2 [\bar{e} \hat{3}][\bar{3} e] \right\} \].
(1.12)

In this expression we have also been able to retain the full dependence on the \( b \)-quark mass since the \( b \) and \( \bar{b} \) momenta only enter the amplitude via overall factors. Upon squaring these produce simple dot products, which we can thus replace in the amplitude by square roots. This is a remarkably simple result which treats the top quarks as on shell, but preserves all the spin correlations. The original calculation is given in ref. [1], which also presents a similar result for the \( gg \rightarrow tt \) process.

1.3 The quark-gluon scattering process

The final example of spinor techniques at tree-level relates to the process shown in Fig. (1.5), \( q\bar{q}gg \) scattering. The momenta are labelled according to,

\[
0 \rightarrow q(p_1) + g(p_2) + g(p_3) + \bar{q}(p_4) .
\] (1.13)

We first decompose the amplitude in terms of colour ordered sub-amplitudes that are separately gauge invariant,
\[
M(p_1, h_1; p_2, h_2; p_3, h_3; p_4, h_4) = g^2 \left( (t^A t^A)_{ii_4} m_1(h_1, h_2, h_3, h_4) + (t^A t^A)_{i_1i_4} m_2(h_1, h_3, h_2, h_4) \right) .
\] (1.14)

This decomposition is clear for the (non-abelian) diagrams (a) and (b) in Fig. (1.5). The diagram containing the triple-gluon vertex enters both \( m_1 \) and \( m_2 \) (with opposite signs) due to the colour algebra relation,
\[
 f^{ABC} t_C = -\frac{i}{\sqrt{2}} [t^A, t^B] .
\] (1.16)

As an aside, we can examine the structure of the squared matrix element by making use of the relation,
\[
 \sum_A t^A_{ij} t^A_{kl} = \delta_{il} \delta_{kj} - \frac{1}{N} \delta_{ij} \delta_{kl} .
\] (1.17)

Evaluating the products of the colour factors in this way we obtain,
\[
 \sum |M|^2 = g^4 \left( \frac{N^2 - 1}{N} \right) \left[ N^2(|m_1|^2 + |m_2|^2) - |m_1 + m_2|^2 \right] .
\] (1.18)
We see that the subleading-in-$N$ squared amplitude is obtained by summing $m_1$ and $m_2$. Hence the triple-gluon vertex diagram cancels and the matrix element is (up to overall couplings) identical to the result obtained for the process $q\bar{q}\gamma\gamma$. This is a generic feature of multigluon scattering amplitudes, with the most subleading contribution in $N$ representing the QED-like diagrams.

We now proceed to the calculation of the diagrams. We take a negative helicity quark line and compute $m_1$ only. Diagram (b) does not contribute, $m^{(b)} = 0$, and the other two diagrams are given by:

\[
m^{(a)} = -i \frac{\langle 1|\epsilon_2+\epsilon_3|4\rangle}{\langle 2|\epsilon_3-\epsilon_2|4\rangle} \left( \langle 1|\epsilon_2|4\rangle + \epsilon_3 \cdot p_2 \langle 1|\epsilon_2|4\rangle - \epsilon_2 \cdot p_3 \langle 1|\epsilon_3|4\rangle \right).
\]

At this point the calculation can be greatly simplified by an astute choice of gauge vectors $b_2$ and $b_3$. When the helicities of the two gluons are the same we shall choose the two reference momenta $b_2 = b_3 = p_1$ so that,

\[
\langle 1|\epsilon_2^+ = \langle 1|\epsilon_3^+ = 0.
\]

We thus see that there is no contribution at all to this amplitude, $m_1(q^-, g^+, g^+, \bar{q}^+) = 0$. Similarly $m_1(q^-, g^-, g^-, \bar{q}^+) = 0$ is easily shown by choosing $b_2 = b_3 = p_4$.

The remaining helicity combination, when the gluons have opposite helicities, is most simple to compute when choosing $b_2 = p_3$ and $b_3 = p_2$. In that case we have the simplification,

\[
\epsilon_2 \cdot \epsilon_3 = \epsilon_2 \cdot p_3 = \epsilon_3 \cdot p_2 = 0.
\]

We again find that the contribution from diagram (c) vanishes and only diagram (a) remains. This is again a remarkable result: we have computed the quark gluon scattering matrix element in a non-Abelian theory, with no net contribution from the diagram involving the three gluon vertex. Its effect is completely fixed by gauge invariance. Completing the calculation we find,

\[
m^{(a)}(1^-_q, 2^+_g, 3^-_g, 4^+_\bar{q}) = i \frac{\langle 13\rangle^2\langle 24\rangle}{\langle 21\rangle\langle 32\rangle\langle 32\rangle}.
\]

By multiplying top and bottom by $\langle 13\rangle$ and using momentum conservation we can put this into a simpler form,

\[
m_1(1^-_q, 2^+_g, 3^-_g, 4^+_\bar{q}) = i \frac{\langle 13\rangle^3}{\langle 12\rangle\langle 23\rangle\langle 14\rangle}.
\]

Similarly, the result for the opposite helicity choice is,

\[
m_1(1^-_q, 2^-_g, 3^+_g, 4^+_\bar{q}) = -i \frac{\langle 34\rangle^2\langle 13\rangle}{\langle 12\rangle\langle 23\rangle\langle 14\rangle}.
\]
Finally, the non-zero amplitudes for $m_2$ can be obtained by Bose symmetry (interchanging 2 and 3),

\[
m_2(1^-, 2^+, 3^+, 4^+) = -i \frac{[24]^2[12]}{[13][32][14]},
\]

\[
m_2(1^-, 2^-, 3^+, 4^+) = i \frac{(12)^3}{(13)(32)(14)},
\]

and parity invariance of the strong interactions means that,

\[
m_i(h_1, h_2, h_3, h_4) = m_i^*(-h_1, -h_2, -h_3, -h_4).
\]

2 Overview of loop calculations

In this section we turn to the issue of calculation of the 1-loop amplitudes that appear at NLO. We review the basic procedure of evaluating loop integrals in dimensional regularization and briefly discuss the traditional approach to performing amplitude calculations.

2.1 Scalar Integrals

\[
I^d_i(m^2_i) = \frac{\mu^{4-d}}{i\pi^{d/2}r_T} \int d^dl \frac{1}{(l^2 - m_i^2 + i\epsilon)},
\]

\[
I^d_2(p_1^2; m_1^2, m_2^2) = \frac{\mu^{4-d}}{i\pi^{d/2}r_T} \int d^dl \frac{1}{(l^2 - m_1^2 + i\epsilon)((l + q_1)^2 - m_2^2 + i\epsilon)},
\]

\[
I^d_3(p_1^2, p_2^2, p_3^2; m_1^2, m_2^2, m_3^2) = \frac{\mu^{4-d}}{i\pi^{d/2}r_T} \times \int d^dl \frac{1}{(l^2 - m_1^2 + i\epsilon)((l + q_1)^2 - m_2^2 + i\epsilon)((l + q_2)^2 - m_3^2 + i\epsilon)},
\]

\[
I^d_4(p_1^2, p_2^2, p_3^2, q_4^2; s_{12}, s_{23}; m_1^2, m_2^2, m_3^2, m_4^2) = \frac{\mu^{4-d}}{i\pi^{d/2}r_T} \times \int d^dl \frac{1}{(l^2 - m_1^2 + i\epsilon)((l + q_1)^2 - m_2^2 + i\epsilon)((l + q_2)^2 - m_3^2 + i\epsilon)((l + q_3)^2 - m_4^2 + i\epsilon)},
\]

(2.1)

where $q_n = \sum_{i=1}^n p_i$ and $q_0 = 0$ and $s_{ij} = (p_i + p_j)^2$. For the purposes of this paper we take the masses in the propagators to be real. Near four dimensions we use $d = 4 - 2\epsilon$. (For clarity the small imaginary part which fixes the analytic continuations is specified by $+i\epsilon$). $\mu$ is a scale introduced so that the integrals preserve their natural dimensions, despite excursions away from $d = 4$. We have removed the overall constant which occurs in $d$-dimensional integrals

\[
r_T \equiv \frac{\Gamma^2(1 - \epsilon)\Gamma(1 + \epsilon)}{\Gamma(1 - 2\epsilon)} = \frac{1}{\Gamma(1 - \epsilon)} + O(\epsilon^3) = 1 - \epsilon\gamma + \epsilon^2 \left[ \frac{\gamma^2}{2} - \frac{\pi^2}{12} \right] + O(\epsilon^3).
\]

(2.2)
Feynman parameter identities are also useful; we have

\[
\frac{1}{A^\alpha B^\beta \ldots F^\phi} = \frac{\Gamma(\alpha + \beta + \ldots \phi)}{\Gamma(\alpha)\Gamma(\beta)\ldots\Gamma(\phi)} \times \int_0^1 da_1 da_2 \ldots da_n \delta(1 - a_1 - a_2 \ldots - a_n) \\
\times \frac{a_1^{\alpha - 1} a_2^{\beta - 1} \ldots a_n^{\phi - 1}}{(Aa_1 + Ba_2 + \ldots + Fa_n)^{\alpha + \beta + \ldots + \phi}}
\]

(2.3)

We shall process this integral using the fundamental formula for one-loop integrals given here,

\[
\frac{1}{i\pi^\frac{d}{2}} \int d^d k \frac{(-k^2)^r}{\left[-k^2 + C - i\varepsilon\right]^m} = [C - i\varepsilon]^{2+r-m-\epsilon} \frac{\Gamma(r + d/2)\Gamma(m - r - 2 + \epsilon)}{\Gamma(d/2)\Gamma(m)}.
\]

(2.4)

After Feynman parametrization and integration over \( d^d k \), we have for the triangle and box integrals,

\[
I_3^d(p_1^2, p_2^2, p_3^2; m_1^2, m_2^2, m_3^2) = -\frac{\mu^{2r}}{r^\Gamma} \prod_{i=1}^3 \int_0^1 da_k \frac{\delta(1 - \sum_j a_j Y_{ij} - i\varepsilon)}{\left[\sum_{i,j} a_i a_j Y_{ij} - i\varepsilon\right]^{1+\epsilon}},
\]

(2.5)

\[
I_4^d(p_1^2, p_2^2, p_3^2, p_4^2; s_{12}, s_{23}, m_1^2, m_2^2, m_3^2, m_4^2) = \frac{\mu^{2r}}{r^\Gamma} \prod_{i=1}^4 \int_0^1 da_k \frac{\delta(1 - \sum_j a_j Y_{ij} - i\varepsilon)}{\left[\sum_{i,j} a_i a_j Y_{ij} - i\varepsilon\right]^{2+\epsilon}},
\]

(2.6)

where \( Y \) is the so-called modified Cayley matrix,

\[
Y_{ij} \equiv \frac{1}{2} \left[ m_i^2 + m_j^2 - (q_{(i-1)} - q_{(j-1)})^2 \right].
\]

(2.7)

### 2.1.1 Dimensional Regularisation

In the intermediate stages of the calculation we must introduce some regularisation procedure to control these divergences. The most effective regulator is the method of dimensional regularisation which continues the dimension of space-time to \( d = 4 - 2\epsilon \) dimensions [2]. This method of regularisation has the advantage that the Ward Identities of the theory are preserved at all stages of the calculation. Integrals over loop momenta are performed in \( d \) dimensions with the help of the following formula,

\[
\int \frac{d^d k}{(2\pi)^d} \frac{(-k^2)^r}{\left[-k^2 + C - i\varepsilon\right]^m} = \frac{i(4\pi)^\epsilon}{16\pi^2} [C - i\varepsilon]^{2+r-m-\epsilon} \frac{\Gamma(r + d/2)\Gamma(m - r - 2 + \epsilon)}{\Gamma(d/2)\Gamma(m)}.
\]

(2.8)
To demonstrate Eq. (2.8), we first perform a Wick rotation of the $k_0$ contour anticlockwise. This is dictated by the $i\varepsilon$ prescription, since for real $C$ the poles coming from the denominator of Eq. (2.8) lie in the second and fourth quadrant of the $k_0$ complex plane as shown in Fig. 2.1. Thus by anti-clockwise rotation we encounter no poles. After rotation by an angle $\pi/2$, the $k_0$ integral runs along the imaginary axis in the $k_0$ plane, $(-i\infty < k_0 < i\infty)$. In order to deal only with real quantities we make the substitution $k_0 = i\kappa_d, k_j = \kappa_j$ for all $j \neq 0$ and introduce $|\kappa| = \sqrt{\kappa_1^2 + \kappa_2^2 \ldots + \kappa_d^2}$. We obtain a $d$-dimensional Euclidean integral which may be written as,

$$\int d^d\kappa \ f(\kappa^2) = \int d|\kappa| \ f(\kappa^2) \ |\kappa|^{d-1} \sin^{d-2} \theta_{d-1} \sin^{d-3} \theta_{d-2} \ldots \times \sin \theta_2 \ d\theta_{d-1} d\theta_{d-2} \ldots d\theta_2 d\theta_1. \quad (2.9)$$

This formula is best proved by induction. The range of the angular integrals is $0 \leq \theta_i \leq \pi$ except for $0 \leq \theta_1 \leq 2\pi$. The angular integrations, which only give an overall factor, can be performed using

$$\int_0^\pi d\theta \ \sin^d \theta = \frac{\Gamma \left( \frac{d+1}{2} \right)}{\Gamma \left( \frac{d+2}{2} \right)} \quad (2.10)$$

We therefore find that the left hand side of Eq. (2.8) can be written as,

$$\frac{2i}{(4\pi)^{d/2} \Gamma \left( \frac{d}{2} \right)} \int_0^\infty d|\kappa| \ |\kappa|^{d+2r-1} \left[ \kappa^2 + C \right]^{-m}. \quad (2.11)$$

This last integral can be reduced to a Beta function, (see Table 2.1)

$$\int_0^\infty dx \ \frac{x^s}{\left[ x^2 + C \right]^m} = \frac{\Gamma \left( \frac{s+1}{2} \right) \Gamma \left( \frac{m-s/2-1/2}{2} \right)}{\Gamma \left( m \right)} C^{s/2+1/2-m} \quad (2.12)$$

which demonstrates Eq. (2.8).
\( \Gamma(z) = \int_0^\infty dt \ e^{-t} t^{z-1} \)
\( \zeta \Gamma(z) = \Gamma(z + 1) \)
\( \Gamma(2z) = \frac{2^{2z-1}}{\sqrt{\pi}} \Gamma(z) \Gamma(z + \frac{1}{2}) \)
\( \Gamma(n + 1) = n! \) for a positive integer
\( \Gamma(1) = 1, \ \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \)
\( \Gamma'(1) = -\gamma_E, \ \gamma_E \approx 0.577215 \)
\( \Gamma''(1) = \gamma_E^2 + \frac{\pi^2}{6} \)

\[ B(a, b) = \int_0^1 dx \ x^{a-1} (1-x)^{b-1} \]
\( B(a, b) = \int_0^\infty dt \ \frac{t^{a-1}}{(1+t)^{a+b}} \) for Re \( a, b > 0 \)
\[ B(a, b) = \frac{\Gamma(a) \Gamma(b)}{\Gamma(a+b)} \]

Table 2.1: Useful properties of the \( \Gamma \) and related functions

2.1.2 Scalar Integrals

Here we give an example of the result a scalar integral regularized by dimensional regularization, \( d = 4 - 2\varepsilon \).

\[ I_4^D (0, 0, 0, 0; s_{12}, s_{23}; 0, 0, 0) = \frac{\mu^{2\varepsilon}}{s_{12}s_{23}} \times \left\{ \frac{2}{\varepsilon^2} \left( (-s_{12})^{-\varepsilon} + (-s_{23})^{-\varepsilon} \right) - \ln^2 \left( \frac{-s_{12}}{-s_{23}} \right) - \pi^2 \right\} + \mathcal{O}(\varepsilon). \] (2.13)

This result is taken from [3]. A basis set of scalar one-loop integrals has been presented in ref. [4]. In addition there is a numerical code, called QCDLoop that returns the numerical value of any one-loop integral as a Laurent series in \( 1/\varepsilon \). Thus the problem of one-loop integrals can be considered as completely solved, at least as far as NLO calculations are concerned.

2.2 Passarino-Veltman reduction

Tensor loop integrals can be reduced to sums of scalar integrals using the Passarino-Veltman decomposition. As an example consider the form factor decomposition of a simple rank 1 triangle diagram.

\[
\int \frac{d^nl}{(2\pi)^n} \left( l^2 - m_1^2 \right) \left( l + p \right)^2 - m_2^2 \right) \left( l + q \right)^2 - m_3^2 \right) = \left( p^\mu \ q^\nu \right) \left( \begin{array}{c} \mathcal{C}_1 \\ \mathcal{C}_2 \end{array} \right) \] (2.14)

\[
\int \frac{d^nl}{(2\pi)^n} \frac{l^\mu l^\nu}{l^2 - m_1^2 \left( l + p \right)^2 - m_2^2 \left( l + q \right)^2 - m_3^2} = \left( p^\mu p^\nu \ q^\mu q^\nu \ (p^\mu q^\nu + q^\mu p^\nu) \ g^\mu\nu \right) \left( \begin{array}{c} \mathcal{C}_{11} \\ \mathcal{C}_{22} \\ \mathcal{C}_{12} \\ \mathcal{C}_{00} \end{array} \right) \] (2.15)
We can solve for $C_1, C_2$ by contracting with the external momenta, $p, q$.

$$
\begin{pmatrix}
R_1 \\
R_2
\end{pmatrix} = \begin{pmatrix}
[2l \cdot p] \\
[2l \cdot q]
\end{pmatrix} = G \begin{pmatrix}
C_1 \\
C_2
\end{pmatrix} \equiv \begin{pmatrix}
2p \cdot p & 2p \cdot q \\
2p \cdot q & 2q \cdot q
\end{pmatrix} \begin{pmatrix}
C_1 \\
C_2
\end{pmatrix}
$$

(2.16)

where the notation is $[2l \cdot p] = \int \frac{d^n l}{(2\pi)^n} \frac{2l \cdot p}{(l+p)^2(l+q)^2}$ by expressing $2l \cdot p, (2l \cdot q)$ as a sum of denominators $2l \cdot p = (l+p)^2 - l^2 - p^2$ we can express $R_1, R_2$ as a sum of scalar integrals

Solving we get

$$
\begin{pmatrix}
C_1 \\
C_2
\end{pmatrix} = G^{-1} \begin{pmatrix}
R_1 \\
R_2
\end{pmatrix}
$$

(2.17)

$G$ is the Gram matrix

$$
G = \begin{pmatrix}
2p \cdot p & 2p \cdot q \\
2p \cdot q & 2q \cdot q
\end{pmatrix}, \quad \Delta_2(p, q) = |G| = 4(p^2 q^2 - (p \cdot q)^2)
$$

(2.18)

$$
G^{-1} = \frac{\begin{pmatrix}
2q \cdot q & -2p \cdot q \\
-2p \cdot q & 2p \cdot p
\end{pmatrix}}{\Delta_2(p, q)}
$$

(2.19)

Thus the solution is $C = G^{-1}R$. This solution appears to have a problem when $p \parallel q$ and the Gram determinant vanishes; the original tensor integral had no special problems when $p \parallel q$. This problem is particularly pervasive in this approach. In the calculation of physical quantities there can be no such singularity and, by appropriate reorganization of terms, they can be explicitly avoided. Their presence greatly complicates calculations performed this way and for a long time this presented the biggest obstacle to computing NLO corrections.

References


