3 Lecture 3: The Thrust distribution in SCET

In this lecture I want to work through one example in detail, and work out the thrust distribution for $\tau \to 0$. I will show the following things:

1. Factorization of the cross-section
2. Matching onto SCET
3. Soft and jet functions
4. RGE and summation of large logarithms
5. Determination of $\alpha_s$

3.1 Factorization of the cross-section

- In this section I want to show that the differential thrust distribution in the limit $\tau \to 0$ can be written as

\[
\frac{1}{\sigma_B} \frac{d\sigma^*}{d\tau} = H_2(Q, \mu) \int d\tau_n d\tau_{\bar{n}} J_n(\tau_n, \mu) J_{\bar{n}}(\tau_{\bar{n}}, \mu) S_2(\tau - \tau_n - \tau_{\bar{n}}, \mu)
\]

- So there are three ingredients: The hard function $H_2$, the jet functions $J_i$ and the soft function $S_2$

- How can we understand/derive this factorization theorem?

- The separation of the hard function from the rest is just the usual EFT matching:

\[
\sum_X |M(e^+e^- \to X)|^2 = |C_{n\bar{n}}|^2 |\langle 0 |O_{n\bar{n}}| X \rangle|^2 = H_2 \sum_X |\langle 0 |O_{n\bar{n}}| X \rangle|^2
\]

- But why does the matrix element $|\langle 0 |O_{n\bar{n}}| X \rangle|^2$ factor into $J_n \otimes J_{\bar{n}} \otimes S_2$?

- While the two different collinear sectors don’t talk to one another directly, (there are no couplings between them in the SCET Lagrangian), soft gluons couple to both of them and mediate interactions.

- This factorization can be understood by introducing yet another Wilson line (a soft Wilson line)

- Define

\[
Y_n(x) = \text{P exp} \left[ ig \int_{-\infty}^{0} ds \ n \cdot A_s(x + ns) \right]
\]
• As all Wilson lines it is unitary $Y_n^\dagger Y_n = 1$ and has the equation of motion

\[ \text{i}n \cdot D_s Y_n = 0, \quad \Rightarrow \quad Y_n^\dagger (\text{i}n \cdot \partial + g n \cdot A_s) Y_n = \text{i}n \cdot \partial \]

• Make a field redefinition of the collinear fields

\[ \xi_n(x) = Y_n(x) \xi_n^{(0)}(x), \quad A_n^a(x) = Y_n(x) A_n^{(0)a} Y_n^\dagger(x) \]

• Using this field redefinition in the Lagrangian, gives

\[ L = \bar{\xi}_n^{(0)}(x) Y_n^\dagger(x) \left[ \text{i}n \cdot D + \frac{1}{\text{i}n \cdot D_n} \text{i}D_n^\dagger \right] \frac{\text{i}}{2} Y_n(x) \xi_n^{(0)}(x) \]

\[ = \bar{\xi}_n^{(0)}(x) \left[ \text{i}n \cdot D_n^{(0)} + \frac{1}{\text{i}n \cdot D_n^{(0)}} \frac{\text{i}}{2} D_n^{(0)} \right] \frac{\text{i}}{2} \]

• Thus, in terms of the redefined fields, there are no more interactions between collinear and soft fields in the Lagrangian

• But now look at the operator mediating the production of the two collinear fermions

• In terms of the redefined filed, it becomes

\[ O_{\bar{n}n} = \bar{\xi}_n W_\bar{n}^\dagger \Gamma W_n \xi_n = \bar{\xi}_n^{(0)} W_\bar{n}^{(0)} \Gamma Y_n W_n^{(0)} \xi_n^{(0)} \]

• The Lagrangian contains no interactions between any of the three types of fields, but the current contains all three of them

• But that means that the matrix element of the operator factorizes simply

\[ \sum_X |\langle 0 | O_{\bar{n}n} | X \rangle|^2 = \langle 0 | O_{\bar{n}n} O_{\bar{n}n}^\dagger | 0 \rangle \]

\[ = \langle 0 | [\bar{\xi}_n^{(0)} W_\bar{n}^{(0)}] \left[ [\bar{\xi}_n^{(0)} W_\bar{n}^{(0)}]^\dagger \right] | 0 \rangle \langle 0 | [W_n^{(0)} \xi_n^{(0)}] \left[ W_n^{(0)} \xi_n^{(0)} \right]^\dagger | 0 \rangle \]

\[ = J_{\bar{n}} \otimes J_n \otimes S_2 \]

• The convolution comes from the fact that all the operators sit at the same point, and the exact form can be derived with a little more effort

\[ \frac{1}{\sigma_B} \frac{d\sigma}{d\tau} = H_{\bar{n}n}(\mu) \int d\tau \int d\tau_s J_{\bar{n}}(\tau_{\bar{n}}, \mu) J_n(\tau_n, \mu) S_{\bar{n}}(\tau_s, \mu) \delta(\tau - \tau_{\bar{n}} - \tau_n - \tau_s) \]
• From this final expression, we can already see one more important other condition for factorization: The observable itself must factorize cleanly in $n$-collinear, $\bar{n}$-collinear and soft contributions.

• For thrust this is easy, since thrust is a linear sum over all particles. Thus

$$\tau = \tau_n + \tau_{\bar{n}} + \tau_s$$

• This is something that has to be considered for each observable

### 3.2 Matching onto SCET

• As discussed before, the matching can be calculated using any "observable" that can be defined in both the full and effective theory

• The easiest way to perform the calculation is the use the matrix element

$$\langle 0 | q \Gamma q | \bar{q} q \rangle \overset{\text{ren}}{=} C_{n\bar{n}}(\mu) \langle 0 | \bar{\xi}_n \Gamma \xi_n | \bar{q}_n q_n \rangle_{\text{ren}}$$

• Both the LHS and RHS are IR divergent quantities (since we can’t observe free quarks), but since SCET reproduces the IR of QCD, all IR divergences should be the same on LHS and RHS.

• To see this, let’s regulate the IR in both theories by putting the quarks off their mass shell ($p_q^2 = p_{\bar{q}}^2 = p^2$)

• At tree level, the matrix elements on the LHS and RHS are one, so that the Wilson coefficient at tree level is

$$C_{n\bar{n}}(\mu) = 1 + \mathcal{O}(\alpha_s)$$

• To calculate the matching at one-loop, we calculate the matrix elements on the LHS and RHS to one loop

• For full QCD, there are three diagrams

• Using dim-reg to regulate the UV and the off-shellness to regulate the IR, we find
– For the QCD vertex diagram [Draw the diagram]

\[
A_{\text{QCD}}^{(1)} = \mu^{4-d} \int \frac{d^dk}{(2\pi)^d} i g \gamma^\alpha T^A \frac{i(p_q + \slashed{k})}{(p_q + k)^2} \gamma_{\mu} - i(p_{\bar{q}} + \slashed{k}) \frac{i(p_{\bar{q}} + \slashed{k})}{(p_{\bar{q}} + k)^2} \gamma_{\alpha} T^A - \frac{i}{k^2} \\
= ig^2 C_F \mu^{4-d} \int \frac{d^dk}{(2\pi)^d} \frac{\gamma^\alpha (p_q + \slashed{k}) \gamma_{\mu} (p_{\bar{q}} + \slashed{k}) \gamma_{\alpha}}{(p_q + k)^2 (p_{\bar{q}} + k)^2 k^2} \\
= -\frac{\alpha_s C_F}{4\pi} \left[ -\frac{1}{\epsilon} + 2 \log \frac{\mu^2}{Q^2} \log \frac{\mu^2}{Q^4} + 2 \log \frac{q^2}{Q^2} + \log \frac{-Q^2}{\mu^2} + \frac{2\pi^2}{3} \right].
\]

where \(Q^2 = 2p_q \cdot p_{\bar{q}} = \bar{n} \cdot p_q n \cdot p_{\bar{q}}\)

– For the collinear diagram [Draw the diagram]

\[
A_{\text{coll}}^{(1)} = \mu^{4-d} \int \frac{d^dk}{(2\pi)^d} i g \left[ n^\alpha + \frac{\gamma_{\perp} \hat{p}_{\perp}}{n \cdot p} + \frac{(\hat{p}_{\perp} + \hat{k}_{\perp}) \gamma_{\alpha}}{n \cdot (p + k)} - \frac{\hat{p}_{\perp} \cdot (\hat{p}_{\perp} + \hat{k}_{\perp})}{n \cdot p n \cdot (p + k)} \right] \frac{i}{2} T^A \\
\times i g^2 C_F \mu^{4-d} \int \frac{d^dk}{(2\pi)^d} \frac{n \cdot \bar{n} \cdot (p + k)}{n \cdot k (p + k)^2 k^2} \\
= -ig^2 C_F \mu^{4-d} \int \frac{d^dk}{(2\pi)^d} \frac{n \cdot \bar{n} \cdot (p + k)}{n \cdot k (p + k)^2 k^2} \\
= -\frac{\alpha_s C_F}{4\pi} \left[ -\frac{2}{\epsilon^2} - \frac{2}{\epsilon} \log \frac{-p^2}{\mu^2} - \log \frac{-p^2}{\mu^2} + 2 \log \frac{-p^2}{\mu^2} - 4 + \frac{\pi^2}{6} \right]
\]

– For the soft graph

\[
A_{\text{soft}}^{(1)} = \mu^{4-d} \int \frac{d^dk}{(2\pi)^d} i g \gamma^\alpha T^A \frac{i n \cdot p_q}{n \cdot p_q n \cdot k + p_{\bar{q}}^2} \frac{\gamma_{\mu}}{n \cdot p_{\bar{q}} n \cdot k + p_{\bar{q}}^2} \frac{i g \gamma_{\alpha}}{T^A} - \frac{i}{k^2} \\
= ig^2 C_F \mu^{4-d} \int \frac{d^dk}{(2\pi)^d} \frac{\gamma_{\alpha} \gamma_{\mu} \gamma^\alpha}{(n \cdot k + p_{\bar{q}}^2 \frac{n}{n \cdot p_{\bar{q}}}) (n \cdot k + p_{\bar{q}}^2 \frac{n}{n \cdot p_{\bar{q}}}) k^2} \\
= -\frac{\alpha_s C_F}{4\pi} \left[ \frac{2}{\epsilon^2} - \frac{2}{\epsilon} \log \frac{-p^2 p_{\bar{q}}^2}{\mu^2 Q^2} \right] \log 2 \log \frac{-p^2 p_{\bar{q}}^2}{\mu^2 Q^2} + \frac{\pi^2}{2}
\]

• The wave function graphs agree between the two theories

• As discussed before, the matching coefficient is the difference between the graphs in QCD and SCET

\[
A_{\text{QCD}}^{(1)} - A_{\text{coll}}^{(1)} - A_{\text{coll}}^{(1)} - A_{\text{soft}}^{(1)} = -\frac{\alpha_s C_F}{4\pi} \left[ \frac{2}{\epsilon^2} + \frac{3}{\epsilon} - \frac{2}{\epsilon} \log \frac{-Q^2}{\mu^2} + \log 2 \log \frac{-Q^2}{\mu^2} - 3 \log \frac{-Q^2}{\mu^2} + 8 - \frac{\pi^2}{6} \right]
\]
• This gives for the $Z$ factors and the renormalized matching coefficients (defining $\lambda = Q/\mu$)

$$
Z_C = 1 + \frac{\alpha_s C_F}{4\pi} \left[ -\frac{2}{\epsilon^2} - \frac{3}{\epsilon} + \frac{2}{\epsilon} \log(-\lambda^2) \right]
$$

$$
C_{n\bar{n}}^{(1)}(Q, \mu) = \frac{\alpha_s C_F}{4\pi} \left[ -\ln^2(-\lambda^2) + 3 \ln(-\lambda^2) - 8 + \frac{\pi^2}{6} \right]
$$
or in terms of the hard coefficient $H = |C|^2$

$$
Z_H = 1 + \frac{\alpha_s C_F}{4\pi} \left[ -\frac{4}{\epsilon^2} - \frac{6}{\epsilon} + \frac{4}{\epsilon} \log \lambda^2 \right]
$$

$$
H_{n\bar{n}}^{(1)}(Q, \mu) = \frac{\alpha_s C_F}{4\pi} \left[ -2 \ln^2 \lambda^2 + 6 \ln \lambda^2 - 16 + \frac{7\pi^2}{3} \right]
$$

• There is one very important check on this calculation and SCET:
  – All IR divergences ($\ln \rho^2$) were reproduced correctly and cancelled in the sum
  – The matching coefficient is IR finite, and only depends on $Q$ and $\mu$

• Since the IR divergences have to cancel by definition, one can do the calculation with any IR regulator one wishes.

• One particularly interesting regulator is to use dim-reg for both the UV and IR.

• One then finds for the three diagrams
  – For the QCD vertex diagram

$$
A_{QCD}^{(1)} = -\frac{\alpha_s C_F}{4\pi} \left[ \frac{2}{\epsilon^2} + \frac{3}{\epsilon} - \frac{2}{\epsilon} \log \frac{-Q^2}{\mu^2} + \log^2 \frac{-Q^2}{\mu^2} - 3 \log \frac{-Q^2}{\mu^2} + 8 - \frac{\pi^2}{6} \right]
$$

  – For the SCET diagrams

$$
A_{\text{coll}}^{(1)} = A_{\text{soft}}^{(1)} = 0
$$

• Thus, the difference between the two theories and the matching coefficient comes out the same, but I only needed to compute QCD diagrams with a much easier regulator
3.3 The soft and jet functions

- The jet and soft functions describe the dynamics of SCET in the collinear and soft sector.

- As indicated in the factorization theorem, they are given in terms of the operators such as
  \[ \langle 0 \left| [Y_n Y^\dagger_n](x) \right| [Y_n Y^\dagger_n]^\dagger(0) \rangle 0 \]  
  and
  \[ \langle 0 \left| [\bar{\xi}_n W^\dagger_n](x) \right| [\bar{\xi}_n W^\dagger_n]^\dagger(0) \rangle 0 \]

- Note that there are products of operators, and not time-ordered products, so we need to figure out how to calculate them.

- One can show that they are related to the discontinuity of a time-ordered product, which can then be related by the optical theorem to the square of a matrix element where we sum over a complete set of states.

- Putting this all together, we find
  \[ S(\tau, \mu) \equiv \frac{1}{N_C} \sum_X \delta \left( \tau - \frac{n \cdot P_X + \bar{n} \cdot P_X}{Q} \right) \left| \langle 0 \left| Y_n X \right| X \rangle \right|^2 \]
  and
  \[ J(\tau, \mu) \equiv \sum_X \delta \left( \tau - \frac{n \cdot P_X}{Q} \right) \left| \langle 0 \left| \bar{\xi}_n W_n \right| X \rangle \right|^2 \]

- At tree level, the calculation is trivial, since the soft final state is empty, while the collinear final state is an on-shell collinear quark with \( n \cdot P_X = 0 \). Thus we get
  \[ S^{(0)}(\tau, \mu) = J^{(1)}(\tau, \mu) = \delta(\tau) \]

- At one-loop, there are virtual and real diagrams. The virtual diagrams are always proportional to \( \delta(\tau) \), but all vanish in pure dim-reg (no scale as before).

- The real diagram for the soft is given by \[ \text{[Draw the diagram]} \]
  \[ S^{(1)}(\tau, \mu) \equiv 2g^2 \mu^{4-d} \frac{d^d k}{(2\pi)^d} \left( \frac{k^+}{k^+ + k^-} \right) \theta(k^-) \times \left[ \theta(k^- - k^+ \delta(\tau - k^+/Q) + \theta(k^+ - k^-) \delta(\tau - k^- /Q) \right] \]
  \[ = \theta(\tau) \frac{\alpha_s C_F}{4\pi} \left( \frac{4\pi \mu^2}{Q^2} \right)^{\epsilon} \frac{1}{\Gamma(1 - \epsilon)} \frac{8}{\epsilon} \left( \frac{1}{\tau} \right)^{1+2\epsilon} \]
  \[ = \theta(\tau) \frac{\alpha_s C_F Q}{4\pi} \frac{1}{\bar{\mu} \Gamma(1 - \epsilon)} \frac{8}{\epsilon} \left( \frac{\bar{\mu}}{Q\tau} \right)^{1+2\epsilon} \]
• How do we expand this in epsilon?
• Naively, we might think that

\[
\left( \frac{1}{x} \right)^{1+2\epsilon} = \frac{1}{x} + 2\epsilon \frac{\ln x}{x} + \ldots
\]

• But this is not quite right, since I can integrate the LHS between zero and 1, but not the RHS

\[
\int_0^1 \left( \frac{1}{x} \right)^{1+2\epsilon} = -\frac{1}{2\epsilon}
\]

but

\[
\int_0^1 \frac{\ln x}{x} = \infty
\]

• But we already know that the soft function is only a distribution (at tree level it was a delta-function).
• The delta-function is defined by how it is integrated

\[
\int dx \delta(x) f(x) = f(0)
\]

• One can define +-distributions, which are defined as

\[
\int_0^1 dx \left[ \ln^n \frac{x}{x} \right]_+ f(x) = \int_0^1 dx \frac{\ln^n x}{x} [f(x) - f(0)]
\]

• In terms of these distributions, one can write

\[
\left( \frac{\theta(x)}{x} \right)^{1+2\epsilon} = -\frac{\delta(x)}{2\epsilon} + \left[ \frac{\theta(x)}{x} \right]_+ - 2\epsilon \left[ \frac{\theta(x) \ln x}{x} \right]_+
\]

• There are a few nice rescaling identities of these distributions

\[
\lambda \delta(\lambda x) = \delta(x),
\]

\[
\lambda \left[ \frac{\theta(\lambda x)}{\lambda x} \right]_+ = \left[ \frac{\theta(x)}{x} \right]_+ + \ln \delta(x),
\]

\[
\left[ \frac{\theta(x) \ln \lambda x}{\lambda x} \right]_+ = \left[ \frac{\theta(x) \ln x}{x} \right]_+ + \ln \lambda \left[ \frac{\theta(x)}{x} \right]_+ + \frac{\ln^2 \lambda}{2} \delta(x)
\]
• Using this, we find for $S^{(1)}$

$$S^{(1)}(\tau, \mu) = \frac{\alpha_s C_F}{4\pi} \frac{1}{\Gamma(1 - \epsilon)} \frac{8}{\epsilon} \lambda \left( \frac{1}{\lambda \tau} \right)^{1+2\epsilon}$$

$$= \frac{\alpha_s C_F}{4\pi} \frac{1}{\Gamma(1 - \epsilon)} \frac{8}{\epsilon} \lambda \left[ -\frac{1}{2\epsilon} \delta(\lambda \tau) + \left[ \frac{\theta(\lambda \tau)}{\lambda \tau} \right] + - 2\epsilon \left[ \frac{\theta(\lambda \tau) \ln \lambda \tau}{\lambda \tau} \right] + \right]$$

$$= \frac{\alpha_s C_F}{4\pi} \lambda \left[ \left( -\frac{4}{\epsilon^2} + \frac{\pi^2}{3} \right) \delta(\lambda \tau) + \frac{8}{\epsilon} \left[ \frac{\theta(\lambda \tau)}{\lambda \tau} \right] + - 16 \left[ \frac{\theta(\lambda \tau) \ln \lambda \tau}{\lambda \tau} \right] + \right]$$

• From this we can read off the Z-factor as well as the renormalized soft function

$$Z_S^{(1)} = \frac{\alpha_s C_F}{4\pi} \lambda \left[ -\frac{4}{\epsilon^2} \delta(\lambda \tau) + \frac{8}{\epsilon} \left[ \frac{\theta(\lambda \tau)}{\lambda \tau} \right] + \right]$$

$$= \frac{\alpha_s C_F}{4\pi} \lambda \left[ \left( -\frac{4}{\epsilon^2} + \frac{4}{\epsilon} \ln \lambda^2 \right) \delta(\tau) + \frac{8}{\epsilon} \left[ \frac{\theta(\tau)}{\tau} \right] + \right]$$

$$S_{\text{ren}}^{(1)} = \frac{\alpha_s C_F}{4\pi} \lambda \left[ \frac{\pi^2}{3} \delta(\lambda \tau) - 16 \left[ \frac{\theta(\lambda \tau) \ln \lambda \tau}{\lambda \tau} \right] + \right]$$

$$= \frac{\alpha_s C_F}{4\pi} \lambda \left[ \left( 2\ln^2 \lambda^2 + \frac{\pi^2}{3} \right) \delta(\tau) - 8 \ln \lambda^2 \left[ \frac{\theta(\tau)}{\tau} \right] + - 16 \left[ \frac{\theta(\tau) \ln \tau}{\tau} \right] + \right]$$

• For the jet function, there are several real diagrams [Draw the diagrams], and we only give the final result

$$Z_J^{(1)} = \frac{\alpha_s C_F}{4\pi} \lambda^2 \left[ \left( \frac{4}{\epsilon^2} + \frac{3}{\epsilon} \right) \delta(\lambda^2 \tau) - \frac{4}{\epsilon} \left[ \frac{\theta(\lambda^2 \tau)}{\lambda^2 \tau} \right] + \right]$$

$$= \frac{\alpha_s C_F}{4\pi} \lambda^2 \left[ \left( \frac{4}{\epsilon^2} - \frac{4}{\epsilon} \ln \lambda^2 + \frac{3}{\epsilon} \right) \delta(\tau) - \frac{4}{\epsilon} \left[ \frac{\theta(\tau)}{\tau} \right] + \right]$$

$$J^{(1)}(\tau, \mu) = \frac{\alpha_s C_F}{4\pi} \lambda^2 \left[ \left( 7 - \pi^2 \right) \delta(\lambda^2 \tau) - 3 \left[ \frac{\theta(\lambda^2 \tau)}{\lambda^2 \tau} \right] + 4 \left[ \frac{\theta(\lambda^2 \tau) \ln(\lambda^2 \tau)}{\lambda^2 \tau} \right] + \right]$$

$$= \frac{\alpha_s C_F}{4\pi} \lambda^2 \left[ \left( 2\ln^2 \lambda^2 - 3 \ln \lambda^2 + 7 - \pi^2 \right) \delta(\tau) - (3 + 4 \ln \lambda^2) \left[ \frac{\theta(\tau)}{\tau} \right] + 4 \left[ \frac{\theta(\tau) \ln \tau}{\tau} \right] + \right]$$

3.4 **RGE and summation of large logs**

• Putting all the information we have obtained so far together, we find

$$\frac{1}{\sigma_B} \frac{d\sigma}{d\tau} = \delta(\tau) + \frac{\alpha_s C_F}{4\pi} \left\{ \left( -2 \ln^2 \lambda^2 + 6 \ln \lambda^2 - 16 + \frac{7\pi^2}{3} \right) \right\}$$
\[+2 \times (2 \ln^2 \lambda^2 - 3 \ln \lambda^2 + 7 - \pi^2)
- \left(2 \ln^2 \lambda^2 - \frac{\pi^2}{3}\right) \delta(\tau)
+ \left(2 \times (-3 - 4 \ln \lambda^2) - 8 \ln \lambda^2\right) \left[\frac{1}{\tau}\right]_+ + \left(2 \times 4 - 16\right) \left[\frac{\ln \tau}{\tau}\right]_+
\] = \delta(\tau) + \frac{\alpha_s C_F}{4\pi} \left\{ -2 + \frac{2\pi^2}{3} \right\} \delta(\tau) - 6 \left[\frac{1}{\tau}\right]_+ - 8 \left[\frac{\ln \tau}{\tau}\right]_+
\]

- Note that as it must, all dependence on \(\mu\) has cancelled once all terms were added together.

- Note also the appearance of the \(\ln \tau\)

- If I were to repeat the same calculation at 2-loops, I would find new terms that go like

\[
\frac{1}{\sigma_B} \frac{d\sigma^{(2)}}{d\tau} \sim \left(\frac{\alpha_s C_F}{4\pi}\right)^2 \left\{ A \delta(\tau) + B \left[\frac{1}{\tau}\right]_+ + C \left[\frac{\ln \tau}{\tau}\right]_+ + D \left[\frac{\ln^2 \tau}{\tau}\right]_+ + E \left[\frac{\ln^3 \tau}{\tau}\right]_+ \right\}
\]

- Each new order in perturbation theory comes with two more factors of \(\ln \tau\)

- So for small enough values of \(\tau\) such that

\[\alpha_s \ln^2 \tau \sim 1\]

we can’t trust the perturbative expansion any more.

- It would be nice to sum up all terms with

\[\alpha_s^n \ln^{2n-1}\]

To get a better behaved perturbative expansion.

- The important thing to notice is that the \(\ln \tau\) come from the jet and the soft function, and they always come in a way defined structure with the renormalization scale.

Jet function: \(\ln(\lambda^2 \tau) = \ln \left(\frac{Q^2 \tau}{\mu^2}\right)\)

Soft function: \(\ln(\lambda \tau) = \ln \left(\frac{Q \tau}{\mu}\right)\)
• This makes sense, since the typical scale in the jet function is the collinear virtuality $\mu^2_J = p^2_c = Q^2 r$, while the typical scale in the soft function is $\mu^2_S = p^2_s = Q^2 r^2$

$$\text{jet function: } \ln \frac{\mu^2}{Q^2 r} = \ln \frac{\mu^2_J}{\mu^2_J}$$

$$\text{soft function: } \ln \frac{\mu^2}{\mu^2_S}$$

• The hard function only depends on logs

$$\text{Hard function: } \ln \frac{\mu^2}{Q^2} = \ln \frac{\mu^2}{\mu^2_H}$$

• Thus, if I were to evaluate the hard, jet and soft functions at the scales $\mu_H$, $\mu_J$ and $\mu_S$, then there are no logarithms in each of the functions.

• But remember that from the factorization theorem one of course has to evaluate each function at the same scale $\mu$

• But can use RGE to write

$$H_{n,\bar{n}}(\mu) = H_{n,\bar{n}}(\mu) U_H(\mu_H, \mu)$$

$$J_{n,\bar{n}}(\mu) = J_{n,\bar{n}}(\mu) \otimes U_J(\mu_H, \mu)$$

$$S_{n,\bar{n}}(\mu) = S_{n,\bar{n}}(\mu) \otimes U_S(\mu_H, \mu)$$

• Given this, we can now rewrite

$$\frac{d\sigma}{d\tau} = H(\mu_H) U_H(\mu_H, \mu) \left[J_n(\mu_J) \otimes U_J(\mu_J, \mu)\right]^2 \otimes S(\mu_S) \otimes U_S(\mu_S, \mu)$$

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