Example 4: Higgs production at NLO
Higgs discovery

Prominent peaks in the $\gamma\gamma$ and $ZZ^* \rightarrow 4l$ modes
What we know so far

- Gross properties of the new state roughly indicate SM-like couplings

- Biggest signals in $\gamma\gamma$ and $ZZ$, which proceed primarily via $gg\rightarrow h$
We showed this plot before indicating that the corrections are large. Our goal now is to compute the NLO cross section for this process and understand why.

Dawson; Djouadi, Graudenz, Spira, Zerwas 1991, 1995
Trouble at NLO

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Without a detailed understanding of QCD, we would have a factor of 3 excess in the $\gamma\gamma$ channel... and even more theoretical frenzy about beyond the SM physics.
Gluon fusion at LO

- Can calculate the LO cross section ⇒ already 1-loop!

\[
\sigma_{gg\to h}^{LO} = \frac{G_F Q_s^2}{288 \pi \sqrt{2}} \left| \sum_Q F_{1/2}(\tau_Q) \right|^2 \delta(1 - z), \quad \tau_Q = \frac{M_H^2}{4m_Q^2}, \quad z = \frac{M_H^2}{\hat{s}}
\]

\[
\tau \to 0 \quad \Rightarrow \quad F_{1/2} \to \frac{4}{3}
\]

\[
\tau \to \infty \quad \Rightarrow \quad F_{1/2} \to -\frac{2m_Q^2}{M_H^2} \ln \frac{M_H^2}{m_Q^2}
\]

- Independent of \( m_f \) when \( m_f \to \infty \) ⇒ true for any heavy fermion that gets its mass entirely from Higgs
Low-energy theorems

Useful, illuminating alternative approach for $2m_t > M_H$

Diagrammatically, clear that Higgs interaction comes from derivatives of the top part of the gluon self-energy:

\[
\mathcal{M}(hgg) \underset{p_H \to 0}{\simeq} \frac{m_t}{v} \frac{\partial}{\partial m_t} \mathcal{M}(gg)
\]

Generates both diagrams in the $M_H \to 0$ limit
Effective field theory

- We’re going to use an effective field theory to calculate the Higgs production cross section.
- EFT: if we are doing experiments at low energies, we shouldn’t care about the dynamics of very heavy particles. We should be able to approximate their effects as local, higher-dimension (suppressed by the heavy-particle masses) operators in an effective Lagrangian.
- Well-established in QCD: heavy-quark EFT, soft-collinear EFT
- We will use the separation $2m_t >> M_H$ to form a Higgs EFT

Useful references on EFT:
Manohar and Wise, *Heavy Quark Effective Theory*
Rothstein, hep-ph/0308266
The Higgs effective Lagrangian

Integrate out the top quark to produce an effective Lagrangian

\[ \mathcal{L}_{full} = -\frac{1}{4} G^a_{\mu\nu} G^{\mu\nu}_a + \mathcal{L}_{top} \]

\[ G^a_{\mu} = \sqrt{\zeta_3} G^a_{\mu} \]

EFT field \hspace{1cm} decoupling constant \hspace{1cm} QCD field

\[ \mathcal{L}_{EFT} = -\frac{\zeta_3}{4} G^a_{\mu\nu} G^{\mu\nu}_a \] (remember to amputate external legs)

Matching calculation: equate full and EFT propagators

\[ -\frac{ig_{\mu\nu}}{p^2} \zeta_3 = -\frac{ig_{\mu\nu}}{p^2} \left[ 1 + \Pi_t(0) \right] \]

\[ \Rightarrow \zeta_3 = 1 + \Pi_t(0) \]

\[ \Rightarrow \mathcal{L}_{EFT} = -\frac{1}{4} \left[ 1 + \Pi_t(0) \right] G^a_{\mu\nu} G^{\mu\nu}_a \] (top-quark contribution to gluon self-energy)
The Higgs effective Lagrangian

Now apply the low energy theorem to derive HGG operator:

\[
\mathcal{L}_{EFT}^{hgg} = - \frac{m_t}{4v} \left( \frac{\partial}{\partial m_t} \Pi_t(0) \right) h G_{\mu\nu}^{a'} G_{a}^{\mu\nu'}
\]

\[
\Rightarrow \Pi_t(0) = \frac{\alpha_s}{6\pi} \left[ \frac{\bar{\mu}^2}{m_t^2} \right]^\varepsilon \frac{\Gamma(1 + \varepsilon)}{\varepsilon}
\]

\[
\Rightarrow \mathcal{L}_{EFT}^{hgg} = \frac{\alpha_s h}{12\pi} v G_{\mu\nu}^{a'} G_{a}^{\mu\nu'}
\]

Numerous nice features of this formulation...
**The Higgs effective Lagrangian**

- Systematically, simply extendable to higher orders in QCD

    Useful references: Kniehl, Spira hep-ph/9505225; Steinhauser hep-ph/0201075

- Reduces calculations by one loop order; 1-loop becomes tree, etc.; makes a NNLO calculation possible

- Turns a two-scale problem into two one-scale problems

Two scales: $M_{\text{Higgs}}, m_{\text{top}}$

\[
\left( \frac{M_{\text{Higgs}}^2}{4m_{\text{top}}^2} \right) \left[ \text{Only } M_{\text{Higgs}} + \text{Only } m_{\text{top}} + \ldots \right]
\]

\[\text{O}(M_{\text{Higgs}}^2/4m_{\text{top}}^2)\]
The Higgs effective Lagrangian

- Factorizes QCD effects (dynamics of gluons, light quarks from L_{EFT}) from new physics (heavy particles into Wilson coefficients)
- Applicable to the $h\gamma\gamma$ coupling also
- Can be used when a particle does not obtain all its mass from the Higgs
- Valid much beyond the expected region of validity; forms the basis for much of Tevatron/LHC phenomenology
- Let’s try it out, and do a full NLO calculation of a hadron collider cross section
Setup

Our Feynman rules are 5-flavor QCD plus the EFT vertices:

\[
\begin{align*}
&= -i \frac{\alpha_s}{3\pi v} \left\{ 1 + \frac{11}{4} \frac{\alpha_s}{\pi} \right\} \delta^{ab} [p_1 \cdot p_2 g^{\mu\nu} - p_1^\nu p_2^\mu] \\
&= g_s \frac{\alpha_s}{3\pi v} f^{abc} \{ g_{\mu\nu} (p_1 - p_2)_\rho \\
&\quad + g_{\nu\rho} (p_2 - p_3)_\mu + (p_3 - p_1)_\nu \}
\end{align*}
\]
Steps

- Pick a regularization scheme (dimensional regularization for us)
- Get the tree-level result
- Calculate 1-loop diagrams as a Laurent series in $\varepsilon$
- Perform the ultraviolet renormalization
- Calculate the real emission diagrams, extract singularities that appear in soft/collinear regions of phase space
- Absorb initial-state collinear singularities into PDFs
- Get numbers
Tree-level

\[ \sigma_{h_1 h_2 \rightarrow h} = \int dx_1 \, dx_2 \, f_g(x_1) \, f_g(x_2) \, \hat{\sigma}(z) \]
+ smaller partonic channels

\[ (z = M_H^2 / x_1 x_2 s) \]

Calculate the spin-, color-averaged matrix element squared

\[ \left| \tilde{M} \right|^2 = \frac{1}{256(1 - \epsilon)^2} \times \left| M \right|^2 = \frac{\hat{s}^2}{576 v^2 (1 - \epsilon)} \left( \frac{\alpha_s}{\pi} \right)^2 \]

Get the phase space and flux factor

\[ \frac{1}{2\hat{s}} \int \frac{d^d p_h}{(2\pi)^d} \, 2\pi \delta(p_H^2 - M_H^2) \, (2\pi)^d \delta^{(d)}(p_1 + p_2 - p_H) = \frac{\pi}{\hat{s}^2} \delta(1 - z) \]
\[ \sigma_{h_1 h_2 \to h} = \int d x_1 \, d x_2 \, f_g(x_1) f_g(x_2) \hat{\sigma}(z) + \text{smaller partonic channels} \]

\[ (z = M_H^2/x_1x_2s) \]

Combine to get the LO result:

\[ \hat{\sigma}_0(z) = \sigma_0 \delta(1 - z) = \frac{\pi}{576v^2} \left( \frac{\alpha_s}{\pi} \right)^2 \delta(1 - z) \]

We will later need the full d-dimensional tree-level result:

\[ \sigma_0^{(d)} = \frac{\sigma_0}{1 - \epsilon} \]
Virtual corrections

Calculate $2 \times \text{Re}[(M_0)^* M_1]$, which appears in the cross section

\[
\sigma_0^{(d)} \frac{\alpha_s}{\pi} \Gamma(1 + \epsilon) \left( \frac{\hat{s}}{\mu^2} \right)^{-\epsilon} \left\{ -\frac{13}{4\epsilon} - \frac{11}{3} \right\} \delta(1 - z)
\]

Leading soft+collinear singularity; emitting gluons from gluons gives color factor $C_A = 3$

External leg corrections scaleless:

\[
\int d^d k (k^2)^n = 0
\]
UV renormalization

LO dependence on $\alpha_S$ gives the UV counterterm:

$$\sigma_0^{(d)} \frac{\alpha_s}{\pi} \frac{1}{\epsilon} \left\{ -\frac{11}{2} + \frac{N_F}{3} \right\}$$

The remaining singularities are of soft/collinear origin; summing what we have so far yields

$$\sigma_0^{(d)} \frac{\alpha_s}{\pi} \left\{ -\frac{3}{\epsilon^2} + \frac{3}{\epsilon} \ln \frac{\hat{s}}{\mu^2} - \frac{1}{\epsilon} \left( \frac{11}{2} - \frac{N_F}{3} \right) + \text{finite} \right\} \delta(1 - z)$$

The pole structure can be checked to be correct: Catani, hep-ph/9802439
Real radiation corrections

Get the corrections coming from emission of an additional gluon

\[ |\tilde{M}|^2 = 24 \alpha_s \sigma_0 \left\{ \frac{(1 - 2\epsilon) M_H^8 + \hat{s}^4 + \hat{t}^4 + \hat{u}^4}{\hat{s}\hat{t}\hat{u}} \right\} \frac{\epsilon}{2(1 - \epsilon)^2} \left( M_H^4 + \hat{s}^2 + \hat{t}^2 + \hat{u}^2 \right)^2 \]

• This can vanish when either \( p_g \to 0 \) (soft), or \( p_g \parallel p_1, p_g \parallel p_2 \) (collinear)
• Need a parameterization of phase space to extract these singularities appropriately

\[ \hat{s} = (p_1 + p_2)^2 \]
\[ \hat{t} = (p_1 - p_g)^2 \]
\[ \hat{u} = (p_2 - p_g)^2 \]
Real radiation corrections

\[
\frac{1}{2\hat{s}} \int \frac{d^d p_g}{(2\pi)^d} \int \frac{d^d p_H}{(2\pi)^d} (2\pi) \delta(p_g^2)(2\pi) \delta(p_H^2 - M_H^2)(2\pi)^d \delta(d)(p_1 + p_2 - p_g - p_H)
\]

Introduce the following parameterization of \( p_g \):

\[
p_g = \frac{\hat{s}(1 - z)}{2} \left( 1, 2\sqrt{\lambda(1 - \lambda)}, 0, 1 - 2\lambda \right)
\]

Obtain:

\[
\frac{1}{16\pi\hat{s}} \left( \frac{s}{4\pi} \right)^{-\epsilon} \frac{1}{\Gamma(1 - \epsilon)} (1 - z)^{1-2\epsilon} \int_0^1 d\lambda [\lambda(1 - \lambda)]^{-\epsilon}
\]

When we combine matrix elements and phase space, get terms of the following form:

\[
(1 - z)^{-1-2\epsilon} \left[ \lambda(1 - \lambda) \right]^{-1-\epsilon}
\]

\( \lambda \to \mathbf{o}, \mathbf{r} \): collinear

\( z \to \mathbf{r} \): soft
Real radiation corrections

The integrals over $\lambda$ can be done in terms of Gamma functions, while the soft singularities as $z \to 1$ can be extracted using plus distributions:

$$(1 - z)^{-1-2\epsilon} = -\frac{1}{2\epsilon} \delta(1 - z) + \left[ \frac{1}{1 - z} \right]_+ - 2\epsilon \left[ \frac{\ln(1 - z)}{1 - z} \right]_+ + O(\epsilon^2)$$

$$\int_0^1 dz \ f(z) \left[ \frac{g(z)}{1 - z} \right]_+ = \int_0^1 dz \ g(z) \left[ f(z) - f(1) \right]$$

Arrive at the following contribution to the cross section:

$$\sigma_0^{(d)} \frac{\alpha_s}{\pi} \Gamma(1 + \epsilon) \left( \frac{\hat{s}}{\mu^2} \right)^{-\epsilon} \left\{ \begin{array}{c}
3 \frac{\alpha_s}{\epsilon^2} \delta(1 - z) - 6 \left[ \frac{1}{1 - z} \right]_+ + \frac{6z(z^2 - z + 2)}{\epsilon} \\
- \frac{3\pi^2}{2} \delta(1 - z) + 12 \left[ \frac{\ln(1 - z)}{1 - z} \right]_+ - 12z(z^2 - z + 2)\ln(1 - z) - \frac{11}{2} (1 - z)^3 \end{array} \right\}$$
Absorb remaining initial-state collinear singularities into PDFs, which amounts to adding the following counterterm:

One for each PDF

\[
2 \times \frac{\alpha_s}{2\pi} \frac{1}{\epsilon} P_{gg} \otimes \hat{\sigma}_0(z)
\]

Arrive at the contribution:

\[
\sigma_0^{(d)} \frac{\alpha_s}{\pi} \frac{1}{\epsilon} \left\{ \left( \frac{11}{2} - \frac{N_F}{3} \right) \delta(1 - z) + \frac{6}{[1-z]^+} - 6z(z^2 - z + 2) \right\}
\]

This cancels all remaining poles, but we need to add on the NLO correction to the Wilson coefficient in the EFT:

\[
\sigma_0^{(d)} \frac{\alpha_s}{\pi} \frac{11}{2} \delta(1 - z)
\]
Final result

Arrive at the final NLO result for the inclusive cross section:

\[
\Delta \sigma = \frac{\alpha_s}{\pi} \sigma_0 \left\{ \left( \frac{11}{2} + \pi^2 \right) \delta(1-z) + 12 \left[ \frac{\ln(1-z)}{1-z} \right]_+ - 12z(-z + z^2 + 2) \ln(1-z) \right\} (M^2/s \leq z \leq 1) \\
- \frac{11}{2} (1-z)^3 + 6 \ln \left( \frac{\hat{s}}{\mu^2} \right) \left[ \frac{1}{[1-z]_+} - z(z^2 - z + 2) \right] \}
\]

First source of large correction: $11/2 + \pi^2 \Rightarrow 50\%$ increase

Second source: shape of PDFs enhances threshold logarithm

\[
\sigma_{had} = \tau \int_\tau^1 dz \frac{\sigma(z)}{z} \mathcal{L} \left( \frac{\tau}{z} \right)
\]

\[
\mathcal{L}(y) = \int_y^1 dx \frac{y}{x} f_1(x) f_2(y/x) \text{ (partonic luminosity)}
\]

Assume $f_i \sim (1-x)^b$; plot $L$ for various $b$

Look for peak near $z=1$

$\Rightarrow$ Sharp fall-off of gluon PDF enhances correction

\[b-2\text{ (valence)}\]
\[b-10\text{ (gluon)}\]
NNLO in the EFT

- Use of the EFT allows the NNLO cross section to be obtained.

Again, scale variation, especially at LO, can badly underestimate error!

Harlander, Kilgore ‘02; Anastasiou, Melnikov ‘02; Ravindran, Smith van Neerven ‘03
Unreasonably effective EFT

NLO in the EFT:

\[ \Delta \sigma = \sigma_0 \frac{\alpha_s}{\pi} \left\{ \left( \frac{11}{2} + \pi^2 \right) \delta(1 - z) + 12 \left[ \frac{\ln(1 - z)}{1 - z} \right] \right\} - 12z(-z + z^2 + 2)\ln(1 - z) \]

\[ -6 \frac{(z^2 + 1 - z)^2}{1 - z} \ln(z) - \frac{11}{2} (1 - z)^3 \}

Identical factors in full theory with \( \sigma_0 \rightarrow \sigma_{\text{LO}}, \) full theory

\[ \sigma_{\text{approx}}^{\text{NLO}} = \left( \frac{\sigma_{\text{EFT}}^{\text{NLO}}}{\sigma_{\text{EFT}}^{\text{LO}}} \right) \sigma_{\text{QCD}}^{\text{LO}} \]

NNLO study of \( 1/m_t \) suppressed operators, matched to large s-hat limit, large indicates this persists

Harlander, Mantler, Marzani, Ozeren; Pak, Rogal, Steinhauser 2009

NLO in the EFT:

analytic continuation to time-like form factor

eikonal emission of soft gluons

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Harlander, Mantler, Marzani, Ozeren; Pak, Rogal, Steinhauser 2009
Summary of gluon fusion

- Serves as a very accurate framework for all LHC phenomenology
- Current uncertainty estimates: roughly 8% from uncalculated higher orders, 7% from PDFs, a few percent from other effects (use of EFT, bottom-quark effects, EW effects)

Useful references: S. Dawson, NPB359 (1991) 283-300 and QCD and Collider Physics by Ellis, Stirling, Webber (detailed NLO calculation); 1101.0593 (detailed discussion of uncertainties)

Available codes: [http://theory.fi.infn.it/grazzini/hcalculators.html](http://theory.fi.infn.it/grazzini/hcalculators.html)
[http://particle.uni-wuppertal.de/harlander/software/ggh@nnlo/](http://particle.uni-wuppertal.de/harlander/software/ggh@nnlo/)