Dimensional regularization

- Several ways to regulate soft/collinear divergences: add a gluon mass, take the quarks off-shell.
- Method of choice is dimensional regularization: work in $d=4-2\varepsilon$ dimensions. Regulate both UV and IR singularities, introduces no new scales in calculations, maintains gauge symmetry.
- Coupling constant becomes dimensionful: $g_s^2 \rightarrow g_s^2 \mu^{2\varepsilon}$

Useful to know the solid angle in $d$-dimensions:

$$\Omega(d) = \frac{2\pi^{d/2}}{\Gamma \left( \frac{d}{2} \right)}$$

$$\int d\Omega(d) = \int dc_\theta \ d\phi \ \left[ s_\theta^2 s_\phi^2 \right]^{-\varepsilon}$$

Infrared singularities regulated via:

$$\int_0^1 dx \ x^{-\varepsilon} \left( \frac{1}{x} \right) = -\frac{1}{\varepsilon}$$
Recompute the phase space and matrix elements for the real radiation corrections

\[ PS_1 \rightarrow PS_0 \times \frac{s}{16\pi^2} \frac{1}{\Gamma(1-\epsilon)} \left[ \frac{s}{4\pi \mu^2} \right]^{-\epsilon} \int dx_1 dx_2 \left[ \frac{(1-x_3)}{(1-x_1)(1-x_2)} \right] \]

also recomputed in d-dimensions

For \( \epsilon \) slightly negative, regulates \( 1/(1-x_{1,2}) \)

\[ |\tilde{M}_1|^2 \rightarrow 2C_F g_s^2 \frac{|\tilde{M}_0|^2}{s} \left\{ \frac{(1-\epsilon)(x_1^2 + x_2^2) + 2\epsilon(1-x_3)}{(1-x_1)(1-x_2)} - 2\epsilon \right\} \]

Combine these to get:

\[ R_{1qgq} = R_0 \times \frac{2g_s^2 C_F}{16\pi^2 \Gamma(1-\epsilon)} \left[ \frac{s}{4\pi \mu^2} \right]^{-\epsilon} \int_0^1 dx_1 \int_{1-x_1}^1 dx_2 \left\{ \frac{(1-\epsilon)(x_1^2 + x_2^2) + 2\epsilon(1-x_3)}{(1-x_1)(1-x_2)} - 2\epsilon \right\} \times [(1-x_1)(1-x_2)(1-x_3)]^{-\epsilon} \]
Final result for real emission

Evaluate integrals (in terms of beta functions) to find:

\[
R_1^{q\bar{q}g} = R_0 \times \frac{\alpha_s C_F}{2\pi \Gamma(1 - \epsilon)} \left[ \frac{s}{4\pi \mu^2} \right]^{-\epsilon} \left\{ \frac{2}{\epsilon^2} + \frac{3}{\epsilon} + \frac{19}{2} - \pi^2 + \mathcal{O}(\epsilon) \right\}
\]

double pole: soft+collinear gluon  
single pole: soft or collinear gluon

Regulator dependent! Not a physical observable.

Add on the virtual corrections next

Derive this expression
Virtual corrections and final result

\[ R_1^{q\bar{q}} = R_0 \times \frac{\alpha_s C_F \Gamma(1 + \epsilon)}{2\pi} \left[ \frac{s}{4\pi \mu^2} \right]^{-\epsilon} \left\{ -\frac{2}{\epsilon^2} - \frac{3}{\epsilon} - 8 + \pi^2 + O(\epsilon) \right\} \]

As required by the KLN theorem, poles cancel upon addition of real and virtual corrections, leaving:

\[ R = R_0 + R_1 + O(\alpha_s^2) = R_0 \times \left\{ 1 + \frac{\alpha_s(\mu)}{\pi} \right\} \]

Derive this expression
Virtual corrections and final result

\[ R_{1}^{\gamma} = R_{0} \times \frac{\alpha_{s} C_{F} \Gamma(1 + \epsilon)}{2\pi} \left[ \frac{s}{4\pi \mu^{2}} \right]^{-\epsilon} \left\{ -\frac{2}{\epsilon^{2}} - \frac{3}{\epsilon} - 8 + \pi^{2} + O(\epsilon) \right\} \]

As required by the KLN theorem, poles cancel upon addition of real and virtual corrections, leaving:

\[ R = R_{0} + R_{1} + O(\alpha_{s}^{2}) = R_{0} \times \left\{ 1 + \frac{\alpha_{s}(\mu)}{\pi} \right\} \]

(A note about scaleless integrals: \( \int d^{d}k \frac{1}{[k^{2}]^{n}} \propto \frac{1}{\epsilon_{UV}} - \frac{1}{\epsilon_{IR}} = 0 \)

Very useful as long as you don’t specifically care about the pole coefficients

Allows us to neglect the external leg corrections)
Renormalization scale (in)dependence

The result must be independent of the arbitrary renormalization scale $\mu$. We can derive the following RG equation:

$$\frac{dR}{d\mu} = 0 \rightarrow \mu^2 \frac{\partial R}{\partial \mu^2} + \beta_{QCD}(\alpha_s) \frac{\partial R}{\partial \alpha_s} = 0$$

Can use this to predict the explicit $\mu$ dependence at higher orders, by expanding this equation as a perturbative expansion in $\alpha_s$

$$\mu^2 \frac{\partial R^{(2)}}{\partial \mu^2} = \frac{\beta_0}{4\pi} \alpha_s^2 \frac{\partial R^{(1)}}{d\alpha_s}$$

$$R^{(2)} = \frac{\beta_0}{4} \left( \frac{\alpha_s}{\pi} \right)^2 R^{(0)} \ln \frac{\mu^2}{s} + \ldots \text{ (\(\mu\) independent)}$$
Variation of scale in some specified range is often used as an estimate of theoretical uncertainty ⇒ if it was calculated to higher orders, this dependence would vanish.

Conventional range: $\sqrt{s}/2 \leq \mu \leq 2\sqrt{s}$

Often underestimates LO→NLO, especially at hadron colliders where qualitatively new effects can appear at higher orders.

How to pick central value with multiple physical scales?

from Keith Ellis
“Theoretical error”

Variation of scale in some specified range is often used as an estimate of theoretical uncertainty ⇒ if it was calculated to higher orders, this dependence would vanish.

LO is a qualitative description at best, and the scale variation is not trustable.

If you want to match the data and have any idea about your error, you need higher orders!
Eikonal approximation

Useful to have diagnostic tools to check pieces of a calculation: ‘eikonal’ approximation for soft gluons gets double pole

\[
\bar{u}^i(p_1) \left[ iM_{0}^{ij} \right] v^j(p_2)
\]

Proportional to the lower-order amplitude, with a color correlation. Emission off the other leg also simplifies
Eikonal approximation

Useful to have diagnostic tools to check pieces of a calculation: ‘eikonal’ approximation for soft gluons gets double pole

\[ = \bar{u}^i(p_1) \left[ iM_0^{ij} \right] v^j(p_2) \]

The real emission amplitude has **factorized** into the tree-level amplitude times an eikonal factor, with non-trivial correlations in color-space, in the soft limit
Eikonal approximation

Phase space also factorizes, into the soft-gluon component times the remainder. Can derive simplified expressions for the cross section in this limit.

\[
d\sigma_s = \left[ \frac{\alpha_s}{2\pi} \frac{\Gamma(1 - \epsilon)}{\Gamma(1 - 2\epsilon)} \left( \frac{4\pi\mu^2}{s_{12}} \right)^\epsilon \right] \sum_{f, f'} d\sigma_{0f f'}^0 \int dS \frac{-p_f \cdot p_{f'}}{p_f \cdot p_s p_{f'} \cdot p_s} dS = \frac{1}{\pi} \left( \frac{4}{s_{12}} \right)^{-\epsilon} \int_0^{\delta_s \sqrt{s_{12}}/2} dE_s \ d\epsilon \ d\theta \ d\phi \ E_s^{1-2\epsilon} s_{\theta}^{-2\epsilon} s_{\phi}^{-2\epsilon}
\]

\[
|\mathcal{M}_{f f'}^0|^2 = \left[ \mathcal{M}_{c_1 \ldots b_f \ldots b_{f'} \ldots c_n} \right]^* T_{b_f d_f}^a T_{b_{f'} d_{f'}}^a \mathcal{M}_{c_1 \ldots d_f \ldots d_{f'} \ldots c_n}
\]

from Harris & Owens hep-ph/0102128, a useful reference for relevant formulae

different expressions depending on soft-particle color representation
Eikonal approximation

Application to the current process yields:

\[
R_{1,soft}^{q\bar{q}g} = R_0 \times \frac{\alpha_s C_F}{\pi} \frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)} \left( \frac{s}{4\pi \mu^2} \right)^{-\epsilon} \left\{ \frac{1}{\epsilon^2} - \frac{2}{\epsilon} \ln \delta + 2 \ln^2 \delta + \text{finite} \right\}
\]

agrees with our full calculation

Cutoff dependence must cancel against other regions of gluon phase space

The $1/\epsilon^2$ poles must cancel against virtual corrections
Collinear approximation

Another singular region to consider: collinear gluon emission. A simple way of calculating this phase-space region also exists. Study the region $p_1 \parallel p_g$. Sudakov parameterization of momenta:

$$p_1^\mu = z p^\mu + k_\perp^\mu - \frac{k_\perp^2}{2 p \cdot n} n^\mu,$$

$$p_g^\mu = (1 - z) p^\mu - k_\perp^\mu - \frac{k_\perp^2}{2 p \cdot n} n^\mu.$$

$k_\perp \rightarrow 0$ is the singular limit. $p$, $n$ are light-like vectors satisfying $p \cdot k_\perp = n \cdot k_\perp = 0$. The amplitude simplifies in this limit:

$$|M_1(p_1, p_2, p_g)|^2 \approx \frac{2}{s_{1g}} g_s^2 \mu^2 \epsilon P_{qq}(z, \epsilon) |M_0(p_1 + p_g, p_2)|^2$$

$$P_{qq}(z, \epsilon) = C_F \left[ \frac{1 + z^2}{1 - z} - \epsilon(1 - z) \right]$$
Collinear approximation

Another singular region to consider: collinear gluon emission. A simple way of calculating this phase-space region also exists. Study the region $p_1 \parallel p_g$. Sudakov parameterization of momenta:

The real emission amplitude has \textit{factorized} into the tree-level amplitude times a splitting function in the soft limit (for gluon splitting, there are correlations in gluon helicity)

$k_\perp \to 0$ is the singular limit. $p, n$ are light-like vectors satisfying $p.k_\perp=n.k_\perp=0$. $p$ bisects $p_1, p_g$. The amplitude simplifies in this limit:

$$|M_1(p_1, p_2, p_g)|^2 \approx \frac{2}{s_{1g}} g_s^2 \mu^2 \varepsilon P_{qq}(z, \varepsilon) |M_0(p_1 + p_g, p_2)|^2$$

$$P_{qq}(z, \varepsilon) = C_F \left[ \frac{1 + z^2}{1 - z} - \varepsilon(1 - z) \right]$$
Collinear approximation

Phase space also simplifies in this limit. We’re left with the following contribution to the NLO R ratio from the $p_1 \parallel p_g$ region:

$$R_{1,1 \parallel g}^{qg} = R_0 \times \frac{\alpha_s}{2\pi} \frac{1}{\Gamma(1 - \epsilon)} \left[ \frac{s}{4\pi\mu^2} \right]^{-\epsilon} \int_{1-\delta_c}^{1} dx_2 \left( 1 - x_2 \right)^{-1 - \epsilon} \int_{0}^{1-\delta} dz \left[ z(1 - z) \right]^{-\epsilon} P_{qq}(z, \epsilon)$$

$$= R_0 \times \frac{\alpha_s}{2\pi} \frac{1}{\Gamma(1 - \epsilon)} \left[ \frac{s}{4\pi\mu^2} \right]^{-\epsilon} \left\{ \frac{1}{\epsilon} \left( \frac{3}{2} + 2 \ln \delta \right) - \ln^2 \delta - \frac{3}{2} \ln \delta_c - 2 \ln \delta \ln \delta_c + \text{finite} \right\}$$

Together with $p_2 \parallel p_g$ region, agrees with full result

Remaining cutoff dependence cancels against hard region of phase space, which is finite and can be handled numerically in 4 dimensions
Slicing and subtraction

- The splitting functions and eikonal factors are universal
- What we’ve done forms the basis of a scheme for handling IR singularities at NLO known as *phase-space slicing*
- Split full=soft+$\Sigma$(collinear)+hard; eikonal+collinear approximations to get singularities
- Numerical integration of hard region; to cancel $\ln(\delta)$, $\ln(\delta_c)$
- Another scheme known as *dipole subtraction*, that unifies the soft and collinear limits into ‘dipoles’ for each pair of emittors
- Another called *FKS* that partitions phase space into regions where only one collinear and one soft singularity can occur

Phase-space slicing, Harris, Owens hep-ph/0102128;  
Dipole subtraction, Catani, Seymour hep-ph/9605323;  
FKS, Frixione, Kunszt, Signer hep-ph/9512328  
Singular limits of matrix elements: Campbell, Glover hep-ph/9710255;  
Catani, Grazzini hep-ph/9908523
Example II: Parton Showers and Jets
The collinear factorization of matrix elements drives the
collection of the parton shower approximation in QCD. Let’s
study this for the case of final-state photon emission from our
\( \gamma^* \) to electrons process just considered (\( C_F \rightarrow I, C_A \rightarrow O \)).

Recall the factorization of the single-emission amplitude in the
\( p_1 \parallel p_g \) limit:

\[
|\mathcal{M}_1(p_1, p_2, p_g)|^2 \approx \frac{2}{s_{1g}} e^2 \mu^2 \epsilon P(z, \epsilon) |\mathcal{M}_0(p_1 + p_g, p_2)|^2
\]

\[
P(z, \epsilon) = \left[ \frac{1 + z^2}{1 - z} - \epsilon(1 - z) \right]
\]

\( z \) = energy fraction carried by final-state fermion.
Constructing the parton shower II

The factorization of the double-real emission matrix element when two photons are collinear to a electron (the triple-collinear splitting function) can be obtained from the literature (Catani, Grazzini hep-ph/9908523)

$$|\mathcal{M}_2(p_1, p_2, p_{g1}, p_{g2})|^2 \approx \frac{4}{s_{1g_1g_2}} e^4 \mu^4 \varepsilon P_2(z_1, z_2 \varepsilon) |\mathcal{M}_0(\tilde{p}_1, p_2)|^2$$

$$\tilde{p}_1 = p_1 + p_{g1} + p_{g2}$$

$$P_2(z_1, z_2) = \left\{ \begin{array}{l}
\frac{s_1 g_1 g_2}{2 s_1 g_1 s_2 g_2} z_3 \left[ \frac{1 + z_3^2}{z_1 z_2} - \varepsilon \frac{z_1^2 + z_2^2}{z_1 z_2} - \varepsilon (1 + \varepsilon) \right] \\
+ \frac{s_1 g_1 g_2}{s_1 g_1} \left[ \frac{z_3 (1 - z_1) + (1 - z_2)^3}{z_1 z_2} + e^2 (1 + z_3) - \varepsilon (z_1^2 + z_1 z_2 + z_2^2) \frac{1 - z_2}{z_1 z_2} \right] \\
+ (1 - \varepsilon) \left[ \varepsilon - (1 - \varepsilon) \frac{s_1 g_2}{s_1 g_1} \right] \right\} + \{g_1 \leftrightarrow g_2\},
\end{array} \right.$$

(17)

Where $z_1, z_2$ = energy fractions carried by final-state photons.

A mess, but we can simplify it...
The strongly-ordered limit

Consider the following picture of the two-emission case

This parton produced with much lower virtuality than the initial one

The intermediate one then splits into a daughter with a still lower virtuality

$S_1 g_2 \ll S_1 g_1 g_2$

The strongly-ordered limit of the collinear splitting
The strongly-order limit

- Make this approximation in the triple-collinear splitting function to obtain:

\[
|\mathcal{M}_2(p_1, p_2, p_{g1}, p_{g2})|^2 \approx \frac{4}{s_{1g1g2} s_{1g2}} e^4 \mu^4 \epsilon P(1 - z_1, \epsilon) P(z_3/(1 - z_1)) |\mathcal{M}_0(\tilde{p}_1, p_2)|^2
\]

Derive this expression

The energy fraction carried by the intermediate fermion after the first splitting

The fraction of energy from the intermediate fermion carried away by the final one

- Reduces to sequential collinear splittings in the strongly-order limit
- After studying the phase space, we will find that this organizes into an exponential of single-collinear splitting
The collinear phase space

We will now combine this with the appropriate phase space for single and double-real emission. Begin with the single emission, which has the following cross section

\[
\sigma_{1\parallel g}^{1} = \frac{1}{2\sqrt{s}} \int \frac{d^dp_1}{(2\pi)^d} 2\pi \delta(p_1^2) \int \frac{d^dp_2}{(2\pi)^d} 2\pi \delta(p_2^2) \int \frac{d^dp_g}{(2\pi)^d} 2\pi \delta(p_g^2) \\
\times (2\pi)^d \delta^{(d)} (p_Z - p_1 - p_2 - p_g) \frac{2e^2}{s_{1g}} |\mathcal{M}_0(p_1 + p_g, p_2)|^2 P(z)
\]

Make the following variable change:

\[
p_1 = \tilde{p}_1 - p_g \\
\frac{d^{d-1}p_1}{E_1} = \frac{\tilde{E}_1}{E_1} \frac{d^{d-1}\tilde{p}_1}{\tilde{E}_1}
\]
We will now combine this with the appropriate phase space for single and double-real emission. Begin with the single emission, which has the following cross section

\[
\sigma^{1\|g}_1 = \frac{1}{2\sqrt{s}} \int \frac{d^d p_1}{(2\pi)^d} 2\pi \delta(p_1^2) \int \frac{d^d p_2}{(2\pi)^d} 2\pi \delta(p_2^2) \int \frac{d^d p_g}{(2\pi)^d} 2\pi \delta(p_g^2) \\
\times (2\pi)^d \delta^{(d)}(p_Z - p_1 - p_2 - p_g) \frac{2e^2}{s_{1g}} |\mathcal{M}_0(p_1 + p_g, p_2)|^2 P(z)
\]

Arrive at the expression:

\[
\sigma^{1\|g}_1 = \sigma_0(p_1, p_2) \int \frac{d^d p_g}{(2\pi)^d} 2\pi \delta(p_g^2) \frac{2e^2}{s_{1g}} \frac{\tilde{E}_1}{E_1} P(z).
\]
Collinear-photon phase space

Begin with the Sudakov decomposition of the momenta

\[ p_1^\mu = z p^\mu + k_\perp^\mu - \frac{k_\perp^2}{z} \frac{n^\mu}{2p \cdot n}, \]
\[ p_g^\mu = (1 - z) p^\mu - k_\perp^\mu - \frac{k_\perp^2}{1 - z} \frac{n^\mu}{2p \cdot n}. \]

\[ \frac{\tilde{E}_1}{E_1 s_{1g}} \frac{d^{d-1} p_g}{E_g} = \frac{d\Omega_{d-2}}{2} dz ds_{1g} [z(1 - z)]^{-\epsilon} s_{1g}^{-1-\epsilon} \]

\[ \sigma_1^{1||g} = \sigma_0(p_1, p_2) \times \frac{\alpha}{2\pi} \left[ \frac{M_Z^2}{4\pi \mu^2} \right]^{-\epsilon} \frac{\Gamma(1 - \epsilon)}{\Gamma(1 - 2\epsilon)} \int \frac{d\phi}{2\pi} \left[ 4s_{2\phi}^2 \right]^{-\epsilon} \]
\[ \times \int \frac{ds_{1g}}{s_{1g}} \left( \frac{M_Z^2}{s_{1g}} \right)^{\epsilon} \int dz [z(1 - z)]^{-\epsilon} P(z) \]

Derive this expression

(N.B.: I assumed \( s^2 = M_Z^2 \))
The double-real emission phase space

We now have to combine the double-real emission matrix elements with the phase space. Begin with:

\[
\sigma_2^{1\|g_1\|g_2} = \frac{1}{2} \sigma_0(\tilde{p}_1, p_2) \times \int \frac{d^{d-1}p_{g_1}}{2E_{g_1}(2\pi)^{d-1}} \int \frac{d^{d-1}p_{g_2}}{2E_{g_2}(2\pi)^{d-1}} \tilde{E}_1 \frac{\tilde{E}_1}{E_1} \\
\times 4e^4 \frac{1}{s_{1g_2}s_{1g_1g_2}} P(1 - z_1)P\left(\frac{z_3}{1 - z_1}\right)
\]

Identical particles in final state

Follow the single-collinear derivation and set:

\[
\frac{d^{d-1}p_{g_1}}{E_{g_1}} = \frac{d\Omega_1}{2} \frac{dk_{T1}^2 k_{T1}^{-2\epsilon} dp_{g_1 z}}{E_{g_1}},
\]

\[
\frac{d^{d-1}p_{g_2}}{E_{g_2}} = \frac{d\Omega_2}{2} \frac{dk_{T2}^2 k_{T2}^{-2\epsilon} dp_{g_2 z}}{E_{g_2}}
\]

Comes from variable change

\[p_1 = \tilde{p}_1 - p_{g_1} - p_{g_2}\]
The double-real emission phase space

To connect $k_{T1}$, $k_{T2}$ to the four-momenta and simplify the measure, keep the following picture in mind:

Introduce a Sudakov decomposition for the first splitting

\[ p_{g_1}^\mu = z_1 \tilde{p}_1^\mu + k_{T1}^\mu - \frac{k_{T1}^2}{z_1} \frac{n^\mu}{2 \tilde{p}_1 \cdot n}, \]

\[ \tilde{p}_{12}^\mu = (1 - z_1) \tilde{p}_1^\mu - k_{T1}^\mu - \frac{k_{T1}^2}{1 - z_1} \frac{n^\mu}{2 \tilde{p}_1 \cdot n}. \]

Intermediate quark momentum; assumed massless here, even though it further splits. This assumes the strongly-ordered approximation.
The double-real emission phase space

To connect $k_{T1}, k_{T2}$ to the four-momenta and simplify the measure, keep the following picture in mind:

Now introduce another such decomposition for the second splitting:

$$p_{g2}^\mu = z'_2 p_{12}^\mu + k_{T2}^\mu - \frac{k_{T2}^2}{z'_2} \frac{n_2^\mu}{2 p_{12} \cdot n_2},$$

$$p_1^\mu = (1 - z'_2) p_{12}^\mu - k_{T2}^\mu - \frac{k_{T2}^2}{1 - z'_2} \frac{n_2^\mu}{2 p_{12} \cdot n_2},$$

$$z'_2 = z_2 / (1 - z_1)$$

Note that the intermediate quark momentum now satisfies:

$$s_{1g2} = (p_1 + p_{g2})^2 = \frac{|k_{T2}^2|}{z'_2 (1 - z'_2)} = \frac{|k_{T2}^2| (1 - z_1)^2}{z_2 z_3}$$
The double-real emission result

Arrive at the following result:

\[
\sigma_2^{1\|g_1\|g_2} = \frac{1}{2} \sigma_0(\tilde{p}_1, p_2) \times \left[ \frac{\alpha}{2\pi} \left( \frac{M_Z^2}{4\pi \mu^2} \right)^{-\epsilon} \frac{\Gamma(1 - \epsilon)}{\Gamma(1 - 2\epsilon)} \right]^2 \int \frac{d\phi_1}{2\pi} [4s_{\phi_1}^2]^{-\epsilon} \\
\times \int \frac{d\phi_2}{2\pi} [4s_{\phi_2}^2]^{-\epsilon} \int \frac{d s_{1g_2}}{s_{1g_2}} \left( \frac{M_Z^2}{s_{1g_2}} \right)^{\epsilon} \int \frac{d s_{1g_1 g_2}}{s_{1g_1 g_2}} \left( \frac{M_Z^2}{s_{1g_1 g_2}} \right)^{\epsilon} \\
\times \int dz_1 dy [z_1(1 - z_1)y(1 - y)]^{-\epsilon} P(z_1)P(y)
\]

Looks messy, but staring closely reveals the following pattern:

\[
\sigma_1 = \sigma_0(\tilde{p}_1, p_2) \times K \\
\sigma_2 = \frac{1}{2} \sigma_0(\tilde{p}_1, p_2) \times K^2
\]

\[\Rightarrow\text{the emission of multiple collinear photons in the strongly-ordered limit exponentiates}\]
The Sudakov form factor

We’ll simplify this exponentiated form for the total cross section (including zero emissions) in several ways

\[ \sigma_{tot}^{1\|g_i} = \sigma_0(\tilde{p}_1, p_2) \times \exp \left\{ \frac{\alpha}{2\pi} \left[ \frac{M_Z^2}{4\pi \mu^2} \right]^{-\epsilon} \frac{\Gamma(1 - \epsilon)}{\Gamma(1 - 2\epsilon)} \int \frac{d\phi}{2\pi} [4s_\phi^2]^{-\epsilon} \right. \\
\times \left. \int \frac{d s_{1g}}{s_{1g}} \left( \frac{M_Z^2}{s_{1g}} \right)^{\epsilon} \int dz [z(1 - z)]^{-\epsilon} P(z) \right\} \]

• Impose a lower cut \( s_{min} \) on the \( s_{1g} \) integration, and cuts on the \( z \) integration. This removes all singularities and allows us to drop the \( \epsilon \) powers.
• Move the exponent to the left-hand side of the equation. This then expresses the no-emission cross section (\( \sigma_0 \)) in terms of the total one.
The Sudakov form factor

Arrive at the expression

\[ \sigma_0(\tilde{p}_1, p_2) = \sigma_{tot} \times \Delta (M_Z^2, s_{min}) \]

\[ \Delta (s_{max}, s_{min}) = \exp \left\{ -\frac{\alpha}{2\pi} \int \frac{d\phi}{2\pi} \int_{s_{min}}^{s_{max}} \frac{ds}{s} \int_{\delta_z}^{1-\delta_z} dz \frac{1 + z^2}{1 - z} \right\} \]

The quantity \( \Delta \) is the **Sudakov form factor**. It gives the probability for emitting no photons from a charged fermion between the scales \( s_{max} \) and \( s_{min} \). We’ve assumed here that \( M_Z \) is the maximum energy available. We will have one such \( \Delta \) for each external leg that can radiate (we’ve only looked at collinear emissions along \( p_1 \) here).

This expression has several useful properties.
Resummation

This expression resums the leading large logarithms associated with restricting radiation in gauge theories. Suppose we are looking at an observable where all photons are restricted to have an energy $E_\gamma < \delta z E_e$, and invariant mass $s_{1g} < s_{\text{min}}$. Can obtain by integrating the Sudakov form factor inclusively.

\[ \Delta \left( M_Z^2, s_{\text{min}} \right) = \exp \left\{ -\frac{\alpha}{2\pi} \ln \frac{M_Z^2}{s_{\text{min}}} \left( \ln \frac{1}{\delta_z^2} - \frac{3}{2} \right) \right\}. \]

Exhibits the famous Sudakov double-log structure. If the cuts are severe, the logs will overwhelm the $\alpha$ suppression, invalidating fixed-order perturbation theory and necessitating the resummation provided by the Sudakov form factor.
The parton shower

We can use the Sudakov form factor to generate exclusive events with multiple photon/gluon radiation in the collinear/strongly-ordered approximation.

- Generate an $s$ according to $\Delta(s_{\text{max}}, s)$. This sets the virtuality of the first branching. If $s < s_{\text{min}}$, no branching occurs.
- Generate $z$ according to the distribution $(1+z^2)/(1-z)$; also generate $\phi$ according to a flat distribution. This gives the 4-momentum of the first emission.
- Generate a new $s'$ according to $\Delta(s, s')$. Repeat procedure until the virtuality falls below $s_{\text{min}}$.

This is the parton shower. Such simulations are heavily used in understanding collider data. Common codes at the LHC are PYTHIA, HERWIG, SHERPA.
The parton shower

- Generates a high multiplicity final state as the shower evolves from high scales down to the lower cutoff

- Much more closely resembles an actual collider event than a fixed-order calculation, which typically only has a few partons emitted into the final state
Limitations

- Important to remember the approximations built into the parton shower. It only describes emissions correctly in the collinear region of phase space. Hard emissions are not described correctly. Can be fixed by matching to exact leading-order matrix elements for hard emissions (CKKW, MLM matching).
- It does not correctly produce the correct normalization from an exact fixed-order calculation. Can be fixed by matching to an NLO result (POWHEG, MC@NLO).
- Only the leading logarithms correctly obtained.

Useful, very complete reference on parton showers and matching: arxiv:1101.2599